Mathematical Beauty in Architecture

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Abstract

Beauty is a universal theme. Architecture provides a different definition of beauty and how to create it compared with mathematical regularities. The fifteen properties of architectural beauty, as described by Christopher Alexander, will be compared with mathematical aspects in order to understand the relevance between the two. According to Alexander, architectural beauty is a result of fifteen interacting properties which are part of architecture. By performing a mathematical analysis on these fifteen properties a more objective description of beauty can be provided.

Alexander's theory

An important participant in the history of architectural theory is Christopher Alexander. He is educated in the fields of Architecture as well as Mathematics. His latest work, "The order of Nature", is an elaborated work to understand the order of nature and the consequences for architecture. In his first book of this series [1] he describes fifteen basic properties necessary to create spaces which contain a certain aspect of 'life'. This can be described as a feature resulting in the appreciation and beauty of spaces and matter. He states that due to the absence of those traditional design tools modern architecture is lifeless in contradiction to many historical buildings. The fifteen properties he uses to support his statement are: 1. level of scale, 2. strong centers, 3. boundaries, 4. alternating repetition, 5. positive space, 6. good shape, 7. local symmetries, 8. deep interlock and ambiguity, 9. contrast, 10. gradients, 11. roughness, 12. echoes, 13. the void, 14. simplicity and inner calm, 15. not-separateness. The cooperation between these 15 properties result in appreciated buildings in which people feel 'comfortable'.

Mathematical beauty

Several mathematicians have defined mathematical beauty. Bertrand Russell, who was one of the most influential mathematicians and philosophers of the twentieth century, expressed his sense of mathematical beauty in these words: "*Mathematics, rightly viewed, possesses not only truth, but supreme beauty -a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry."* [2].

Considering Russell's quotation regularity, logic and distinctiveness are the fundamental properties. According to Pythagoras, the mysticism of natural numbers determines the beauty of mathematics. All these aspects are of major importance considering mathematical beauty.

Comparison descriptions of beauty

When comparing the architectural definition of beauty with mathematics this will result in a better understanding of the relationship between the two. Therefore, the fifteen properties as stated by Alexander are being clarified by using mathematical principles.

1. level of scale. Buildings are constructed by putting smaller and bigger parts together. All these parts have influence on the perception of the building. According to Alexander each used part has an own centre. [1] This topic will be elaborated in the following paragraph about strong centers. Within the total assembly the level of scale and scale differences determines the tension between the centers and contributes to the 'life' of the building. The property of having levels of scale is not a mechanical thing which can require a wide range of different scales, but 'life' can only occur when each center gives life to the next one. In order to create a good composition, Alexander states that one jump in scale should be 1:3. So, when a larger jump of scale is necessary several steps with the ratio 1:3 should be taken, because else the relation between scale levels too hard to be noticed. A too large scale jump does not do anything to bring life to a structure.
Note that this ratio seems to be closely related to the natural number $e = 2.718$", [3] as well as $\pi = 3.14$". The irrational number e is sometimes called Euler's number because he was the first one to discover this natural logarithm. Both e and π are constants within the mathematical system. When using ratio 1:e to create one scale jump the human brain is still able to notice and understand the relation between two levels of scale, yet the human fails to notice the relation and scaling when the ratio is larger, for example 1:20 as shown in figure 1.

Figure 1: *Relationship between ratio 1:e (left) and 1:20 (right)*

2. strong centers. The frequent appearance of strong centers in structures is another property which brings 'life' to a structure. In his theory, Alexander[1] states that centers play a much more fundamental role as basic element to create architectural beauty within a composition compared to the other fourteen properties. A strong centre appears when different layers are put together to create a composition. In this way a vector field is set up. Every point within the total assembly has the property that from that point the centre of a field is in a certain direction: one direction is going to the centre and another direction is moving away from the centre. As a result, the whole visual field is oriented towards the centre and the field appears to be centered. The hierarchy of the layers creates a deep feeling and intensity of the centre. Though, every layer needs to be a 'structure' which forms centers itself. The arising of a gradient of increasing intensity creates a centre in the middle. So, the field effect and the power of a centre are created by the sequence of other nearby centers leading up to it. It is this multiplicity of different centers, at different levels, that engages human beings by feeling the presence of a centre.
When concerning the mathematical definition of strong centers, this is strongly related to the determination of the centre of gravity of a specific field. [4] The location of this centre depends on the shape and form of the field. Figure 2 shows the determination of the centre of gravity of a basic geometry. When the field is divided in multiple parts will create new centers of gravity for each individual part. The total centre of gravity can now be determined by the combination of these multiple

individual centers. When the multiple parts are forming an irregular subdivision it is possible that the centre of gravity of the total field will move away from its original point. This effect is shown in the right diagram of figure 2.

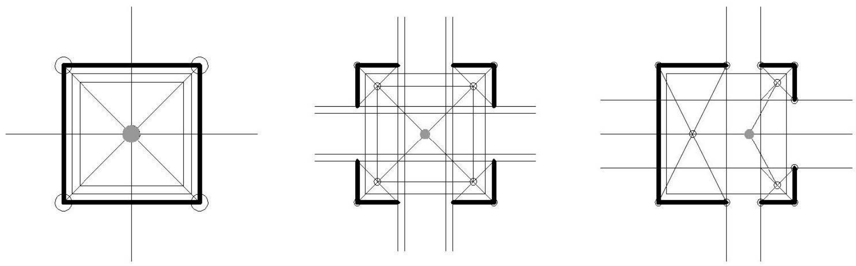

Figure 2: Determination of the centre of gravity

3. boundaries. Living centers are often formed and strengthened when they are surrounded by boundaries. A boundary is forming a field of force around a centre and in this way focuses the attention on the centre of a space. It helps to produce and intensify the centre which it is bounding. At the same time a boundary unites the centre which is being bounded with the world at the other side of the boundary. The boundary thus separate as well as unites a centre with the space beyond the boundary. In both ways the centre becomes more intense. According to Alexander a boundary needs to be of the same order of magnitude as the centre which is being bounded. He also states that any centre which has 'life', like a building or a town, must have a boundary. So, without boundaries the strong centers, which are described in the previous section, could not exist. This is shown in figure 2, where cuts in the boundaries determine the changes in center points.
Considering mathematics, boundaries are strongly related to geometrical forms. These start with a single point. The relation between two points can be defined as a line. A line can be further developed by implementing another point so that a triangular shape is created. This triangle transforms to a rectangle which transforms into a polygon by implementing new points within the total composition. The final form can be related to the base shape of the circle. So every bounding area can be related to this circular base form. This effect is shown in figure 3.

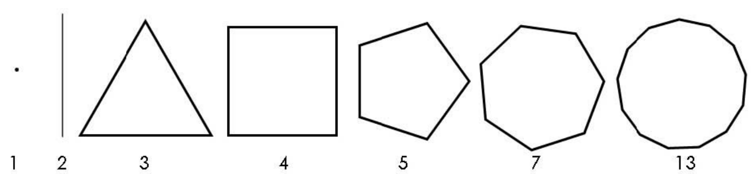

Figure 3: Polygon boundaries

4. alternating repetition. The relation between centers can be intensified by the use of repetition. A certain rhythm of a repeating centre intensifies the field effect of the total composition. Though, this effect is not just a simple repetition. There are two basic characteristics when concerning repetition: the basic element and a defined action. Alexander states that when a single element is literally being repeated, a boring image appears. He thinks that the element should alternate in order to create 'life' in a composition. The effect of this difference is shown in figure 4. In the upper series the element is being repeated by one and the same action over and over again. The two lower series are undergoing an alternating action which results in a change of the element's shape.
Compared to mathematics this will result in a series that alternates by undergoing the action. For example the series of $f_n = f_{n-1}+1$ with $f_0 = 0$, which results in the series 0, 1, 2, 3, 4, 5, … n+1. This is a much more interesting series than $f_n = f_{n-1}$, resulting in the series with an equal result to f_0.

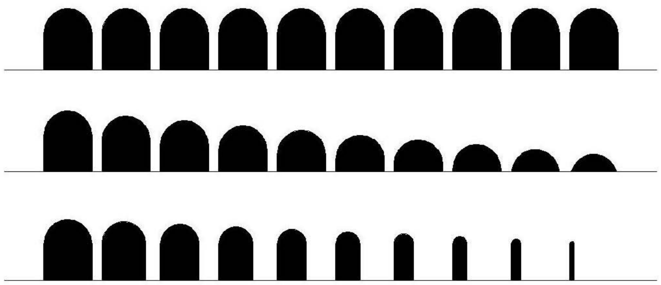

Figure 4: Alteration of the element by undergoing an action, the top diagram shows a non-alternating repetition ($f_n = f_{n-1}$), the middle and bottom show different actions of change like lowering and narrowing ($f_n = f_{n-1}$+action).

5. positive space. The void which results from the positioning of mass has its own identity due to its usefulness and the relationship with the surrounding mass. Considering architectural beauty, positive space is difficult to define.
When reviewing positive space in a two dimensional way, positive space can be the result of translation, rotation and minimal transformation of the negative space: the mass. This effect is shown in figure 5. Alexander states that the interaction between the mass and the void results in the fact that every bit of space is very intensely useful and that there is no leftover space which is not useful. So a positive space has a very strong geometrical relation with the mass from which the space is the resultant.

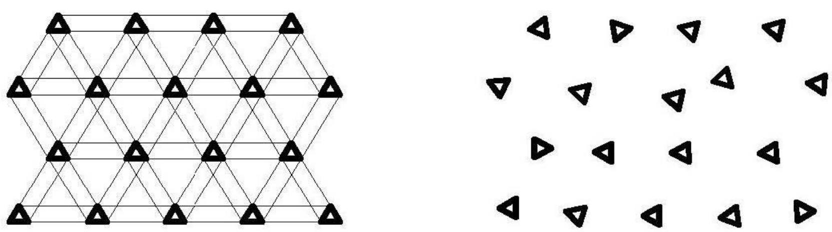

Figure 5: The resulting space with a strong relation to the mass (left) and without (right)

6. good shape. A shape can be built up from elements which all have strong centers. Every element should have its meaning for the final goal which the total composition wants to achieve. The centers of the several elements are in this way very important, because they affect, support and strengthen each other in order to create a good shape. When elements are missing or dislocated within the total arrangement, this has a negative effect on the final composition.
Mathematically this creates a problem, because the simple sum of all elements is not necessarily the right way to achieve a maximal result. This means that the sum of all elements should result in a bigger outcome than the expectation. Mathematically this is not possible ($1+1\neq3$) and it is therefore impossible to define good shape in an objective manner.

7. local symmetries. According to the theory of Christopher Alexander [1], 'life' can only occur when symmetry is present on several levels of scale. Symmetry in only one specific element is not enough to bring 'life' in general. It only appears when several elements, which all have an own symmetrical axis, are placed together to form a total composition. By using symmetry, new and stronger centers occur which brings 'life' to the whole.
This mirror effect is a result of mathematical beauty which is very easily recognized by the human brain. The symmetry of elements attracts the human eye to the central point where the mirroring takes place, so that the new centre is being highlighted. Mirroring can be done in relation to a point, line or surface. This effect is shown in figure 6 in which a simple line is mirrored two times and creates a strong composition.

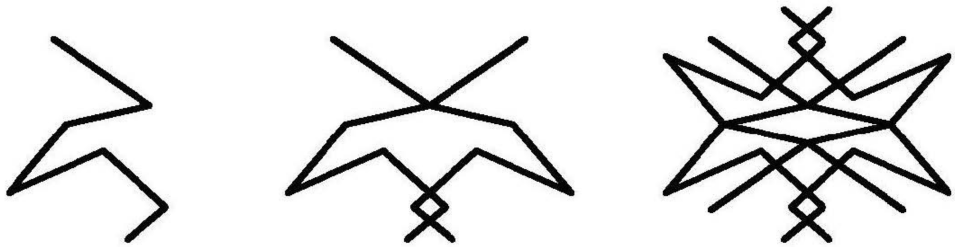

Figure 6: Creation of a stronger image by using symmetry

8. deep interlock and ambiguity. This property contains the interrelation between two or more elements which react on each others shape to create a new unity. Space that occurs between the elements forms a new centre and gets a function as mediator between the elements of the new whole. They also react on their direct surroundings so that the new unity is able of being adapted by its environment.
In a psychological way of thinking it is possible to state that the interrelation between different elements can refer to mathematical symbols which are used to calculate with numbers. These symbols are also the mediators between the elements, the numbers, of math to create a new outcome. So when the property of deep interlock and ambiguity is considered in a more objective way, mathematics is not able to give a proper definition.

9. contrast. When several elements of different shape, texture or color are combined into one composition, their contrast results in a more comprehensive and unified whole. The contrast within a composition will be strengthened when the specifications of the elements differ more extremely like an image with only black and white surfaces is a simplified image. Though, more differentiation between the elements generates a more complicated image in which each individual effect on the total image gets harder to understand. Like a grayscale image, showing more detail, but making the image more complex. A clear contrast between the participating elements should create and allows a clear differentiation in order to create a unified image. It can be stated that there is a relation between contrast and the mathematical subject of equations. Equations are compositions in which the participating elements all have their own function in order to create a result. When the elements of an equation have a clear relation to each other this will also result in a clear perception of the total function. For example the combination of the contrasting functions $f_{(1)} = x^2$, $f_{(2)} = 400$, which gives $f_{(1,2)} = x^2 + 400$ as shown in figure 7a. The relation between the elements of $f_{(1)}$ and $f_{(2)}$ gives a clear perception of their influence on the final outcome $f_{(1,2)}$. When the function gets a more complicated character, the influence of each element within the total function will also be more difficult to understand. For example the combination of the contrasting functions $f_{(1)} = 1/3\ x^2$, $f_{(2)} = \sqrt{(3x)}$, $f_{(3)} = 300\sin(x)$, $f_{(4)} = 400$, $f_{(5)} = x^2$, which gives $f_{(1,2,3,4,5)} = 1/3\ x^2 - \sqrt{(3x)} - 300\sin(x) + 400 + x^2$, as shown in figure 7b. Because of the more differentiated character of this function, the relation and the effect of the participating elements on the whole is much harder to understand.

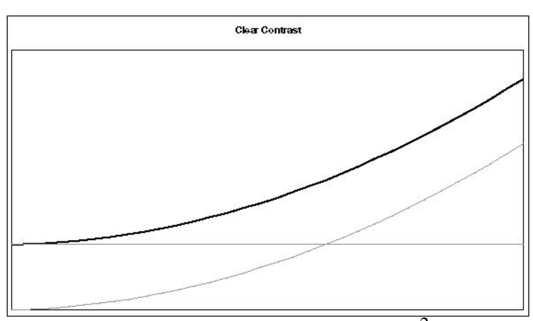

Figure 7a: Relation in the function: $f_{(1,2)} = x^2 + 400$ (dark line)

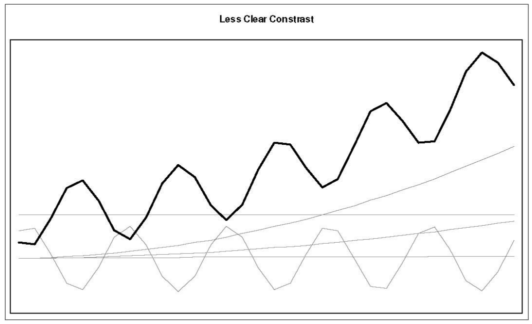

Figure 7b: Relation in the function:
$f_{(1,2,3,4)} = 1/3x^2 - \sqrt{(3x)} - 300\sin(x) + 400 + x^2$ (dark line)

10. gradients. Qualities vary, slowly, subtly, gradually, across the extent of each thing. This gradual change is comparable with series as described in the paragraph concerning alternating repetition. A gradient is a mediator, which slowly changes of appearance in a certain direction and with a certain regularity.
Considering mathematics, a gradient shows remarkable similarity with mathematical functions. The mathematical concept of a function expresses dependence between two quantities; one which is given by the input which produces the output. Of major importance considering this function is that it should not be constant, but gradually changes. The function itself determines the rate, intensity and direction of the gradient. The tangent of the function is very important in defining these characteristics of the gradient at a certain position. When the tangent of a function is not zero, the related direction of the function describes a gradient.

11. roughness. According to Christopher Alexander [1], roughness can be defined as a pre-defined grid which has certain imperfections which are related to the pattern. These imperfections generate a local disturbance within an image and they are thus developing a certain tension in the total field. In this way, the imperfections draw the attention of the spectator and bring 'life' to a composition. The effect of imperfections within a field is shown in figure 8 and 9.
Compared to mathematics, this effect can be related to a normal deviation in which most points are very near to the average, with some exceptions. The curve of the deviation determines the amount of irregularities. A wide and low curve contains many irregularities yet a high and narrow curve shows minimal irregularities. It is impossible to determine a deviation to define the aliveness of roughness; this is a matter of perception strongly related to the specific situation. Factor playing a role are: diversity, density, grid structure and amount and strength of anomalies.

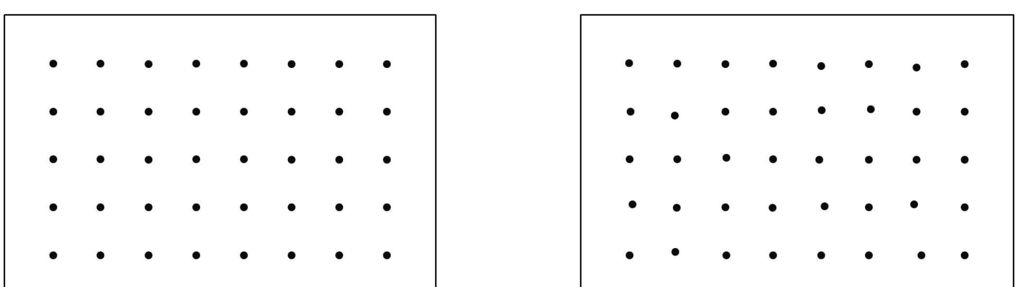

Figure 8: Disturbance of the strict grid (left) by dislocation of certain points (right) resulting in a more lively composition

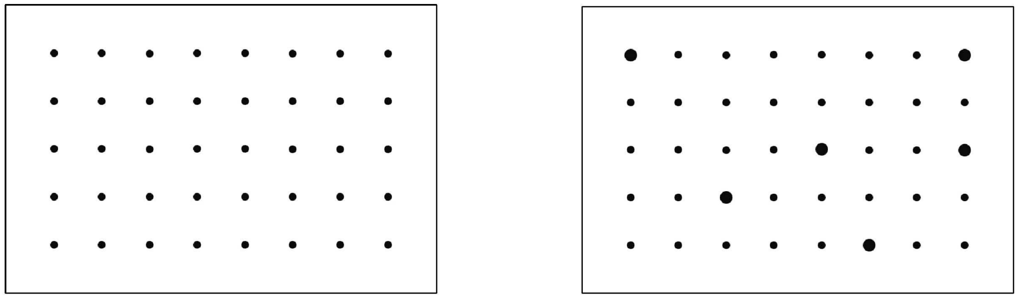

Figure 9: Disturbance of the strict grid (left) by scaling of certain points (right) resulting in a more lively composition

12. echoes. An echo can be defined as a repetition of a base object with a possible alteration. An important aspect considering echoes is the geometrical reference to a base shape. Alexander states that echoes depends on angels and families of angles which are prevalent in a design. In general terms, there is a deep underlying similarity among the elements of a composition which generates the perception that everything seems to be related.
The relation between echoing elements in a larger whole can be the result of a transformation by mathematical aspects like translation, scaling and rotation or a combination of them as shown in figure 10. Note that every transformation results in the same geometrical shapes as present in the previous one.

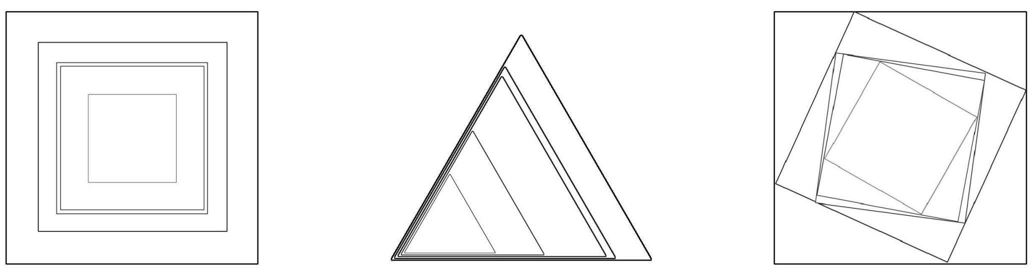

Figure 10: Repetition by scaling (left), scaling & translation (middle) and scaling & rotation (right)

13. the void. Void is used to develop activity by the usage of empty space. Surrounding structures form inner spaces and the absence of objects result in a space with the opportunity to develop activities. Void is necessary in every building, it results in a place of rest and calm. The surrounding fabric determines the feeling of the void, yet it is important that the void has enough space to exist and to be used. It is impossible to relate this property to a mathematical aspect.

14. simplicity and inner calm. In order to create understandable architecture the use of very simple, geometrical shapes is necessary. The proportions of these geometrical shapes need to be unusual. Therefore, the shapes are a result of an unusual parameter like length, size, width, etc. It is also important that there are little imperfections within the total composition, although they are necessary to create a living whole as explained in the paragraph about roughness.
Mathematically this means the use of understandable shapes, like squares and triangles, yet they should have one unusual parameter which will determine the composition. An example which shows the result of simplicity and inner calm is shown in figure 11.

Figure 11: Without inner calm (left) and with inner calm (right)

15. not-separateness. A shape can be considered not-separated, when it is experienced as one with the world and not separated from it. This means that any centre which has 'deep life' is connected, in feeling, to what surrounds it, and is not cut off, isolated, or separated. According to Alexander [1], this last property about not-separateness is the most important property of all properties as described before. He states that a shape cannot have 'life' when it is not unified with its surroundings, even though it should contain the other fourteen properties in it. Considering not-separateness in mathematics, this property cannot be defined in an objective way.

Conclusion

By the mathematical analysis of Christopher Alexander's properties of 'life' a more comprehensive image of his theory is created. The attempt of objective description has showed on the one hand some interesting similarities yet also shows that mathematics by itself is not capable to define architectural beauty completely. The presence of natural constants like π and e in architecture has been noticed in earlier research but still it is important to notice the relation by these important mathematical numbers in human perception. How is it possible that these numbers are comprehensive and comfortable? Architectural beauty is strongly related to geometry yet with the addition of (mathematical) functions of change and irregularity. Mathematicians generally state that a uniform description of a function which relates to the heart of the problem or case is a beautiful one, yet human appreciation and comfort seems also highly related to irregularities and anomalies within a comprehensive order. According to Alexander contemporary architecture moves away from the usage of his fifteen traditional building design tools. This research can be seen as an opportunity to test contemporary architecture in a more objective and mathematical way and reveal the differences between historical and modern buildings. This paper provides a more technical approach on Alexander's fifteen properties and the relation between architectural and mathematical beauty.

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